Brownian Motion in a Magnetic Field*†

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Diffusion of electrons and ions in a plasma across a magnetic field with the interplay of an anisotropic dynamical friction is formulated in terms of the methods of Brownian motion. For a special symmetry of the dynamical friction matrix it is found that for Larmor periods of the order of the relaxation time across the magnetic field the diffusion takes place as an ordinary Brownian motion uninhibited by the external magnetic field.

I. INTRODUCTION

IN a previous paper[1] the author has discussed the diffusion process in a plasma as a Brownian motion arising from local fluctuating electric fields in the plasma. However, the formulation given there leads to the use of a distribution operator for the calculation of the average values of various quantities. From a classical point of view the experimental observation of a distribution operator cannot be defined unambiguously. It is, therefore, necessary to give a general formulation of the problem based on a more conventional discussion of Brownian motion. Furthermore, the assumption of spherical symmetry for the distribution functions for the diffusion of charges in a fixed magnetic field suffers from certain drawbacks. The spherical symmetry of the distribution functions in momentum or configuration spaces is closely related to the assumption that the dynamical friction of charges is a scalar.

A more natural description can be found by noting that a plasma placed in an external magnetic field is not a spherically symmetric system and, therefore, the dynamical friction cannot be independent of the possible anisotropic distribution of momenta in such a plasma. In a plasma one takes into account the effect of collisions between particles by means of fluctuating local electric fields which influence the motions of the particles in the manner of Brownian motion. One of the basic differences between Boltzmann and Brownian motion description lies in the fact that the latter expresses most of the dynamical properties of the system in friction coefficients instead of by direct use of collisions cross sections as in the former approach.

The interparticle interaction does not change the Brownian character of the motion since the actual value of the electric field at any particular instant will depend on the instantaneous position of all other particles and is, therefore, subject to fluctuations. The processes of ionization and recombination can also contribute to the fluctuation of electric field. Because of these facts one cannot obtain the exact dependence of the field on the position or time. However, one can calculate the probability of a given electric field strength at a point in a plasma. This problem has been treated by Holtzmark.[2]

What is important in this case is the cumulative effect of a large number of separate events each of which has only a very minute effect. The total sum of these effects, lasting about a time interval Δt, say, produces an appreciable change in the momentum of a Brownian particle. The resulting motion is analogous to a random walk problem. The random walk of the charge, in this case, is caused by an anisotropic dynamical friction superposed on local random fluctuations of electric field. The anisotropy in question can be established by observing that the external magnetic field has a definite influence on the hyperbolic orbit of a charge obtained upon a collision with another particle. The magnetic field will cause the orbits to rotate. The resulting increments in the momentum of the particle cannot be resolved into two components parallel and perpendicular to the initial direction of motion. This means that the relation between the average and initial momenta is not simple.

A particle deviates from its initial state at different rates in different directions. Thus, we shall assume that a dissipative force of the form

$$F_f = -f|p\rangle \tag{I.1}$$

will operate, where the dynamical friction matrix f is a 3×3 symmetric matrix and the symbol $|p\rangle$ represents a column vector:

$$|p\rangle = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} \beta & \gamma & \delta \\ \gamma & \mu & \rho \\ \delta & \rho & \nu \end{pmatrix}, \tag{I.2}$$

and where the eigenvalues of f are real. In order for an equilibrium state to be approached, the six independent elements of f must be restricted by the statements

$$\begin{array}{ll} \text{(i)} & \text{trace } f>0; \\ \text{(ii)} & \beta\mu-\gamma^2>0, \quad \mu\nu-\rho^2>0, \quad \beta\nu-\delta^2>0; \\ \text{(iii)} & \det\ f>0. \end{array} \tag{I.3}$$

These conditions are sufficient for f to have positive eigenvalues.

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[1] B. Kurşunoglu, Ann. Phys. (N. Y.) 17, 259 (1962).

[2] A. Holtzmark, Ann. Physik 58, 577 (1919).

In the simple case of a scalar dynamical friction it is rather simple to evaluate its general form. In this case β arises from a systematic tendency of the particle to be decelerated in the direction of its motion by an amount proportional to $|v|$. We shall use Chandrasekhar's[3] formula for dynamical friction calculated for a star acted on by fluctuating gravitational fields. We replace the constant of gravitation G by e^2/mM and the average distance between stars by the Debye length and obtain for the scalar dynamical friction coefficient β, for a plasma with Maxwellian distribution, the result

$$\beta=\frac{8n\sqrt{\pi}}{M}\frac{e^4}{3\kappa T}\left(\frac{m}{3\kappa T}\right)^{1/2}[\Phi(x_0)-x_0\Phi'(x_0)]\ln X, \quad (I.4)$$

where

$$\Phi(x)=\int_0^x e^{-x^2}dx,$$

$$x_0=j(3\kappa T/m)^{1/2},$$

n = average number of particles per unit volume, and j is a parameter which measures the dispersion of velocities in the system. The quantity X is defined as

$$X=\langle v^2\rangle\frac{\lambda_D mM}{e^2(m+M)}, \quad \lambda_D=\left(\frac{\kappa T}{4\pi ne^2}\right)^{1/2}. \quad (I.5)$$

A method similar to that for the gravitational case used by Chandrasekhar can be developed for the calculation of a friction matrix. The fluctuation of the electromagnetic field can be analyzed in terms of individual two-particle collisions where each is represented as a two-body problem. Because of the magnetic field the net change in velocity cannot be resolved into perpendicular and parallel components and, therefore, calculation of the various components of the friction tensor f_{ij} will be quite complicated. The actual computation of f_{ij} will not be important for purely qualitative discussions in this paper.

At this point we would like to remark that a more realistic picture, for the stochastic processes in plasma, must take into consideration the possibility of existence of randomly fluctuating magnetic fields in the plasma. Such a possibility may give rise to a new mechanism effecting the diffusion of particles across magnetic fields with actual exchange of energy between particles and fields in the plasma. The solution of this problem would require a combined study of Maxwell's equations and generalized Langevin's equation where Maxwell's equations are to be regarded as stochastic equations, the source of the field being an external current superimposed over a fluctuating internal current of the plasma. In this case solutions of Maxwell's equations, as equations for the irreversibly fluctuating electric and magnetic fields, will consist of the statistical properties of $\mathcal{E}(t)$ and $B(t)$. Such a problem for gravitational field has been discussed in detail by Chandrasekhar and von Neumann.[4]

II. GENERALIZED LANGEVIN EQUATION

By introducing a dynamical friction matrix f, we modify Lorentz's equations of motion into the form

$$\frac{d}{dt}|p\rangle=-\Lambda|p\rangle+|F(t)\rangle, \quad (II.1)$$

where, in the absence of a fluctuating magnetic field, we have

$$\Lambda=f-i\omega_c K_3, \quad K_3=\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$|F(t)\rangle=e|\mathcal{E}(t)\rangle=\text{fluctuating electric force},$$

and f is defined by (I.2). The magnetic field is taken in the Z direction.

The formal solution of the stochastic differential equation is

$$|Y\rangle\equiv|p\rangle-e^{-\Lambda t}|p_0\rangle=\int_0^t e^{\Lambda(\xi-t)}|F(\xi)\rangle d\xi. \quad (II.2)$$

Now, the left-hand side of (II.2) must have the same statistical properties as its right-hand side. We shall generalize the method used in Chandrasekhar's paper[5] to the present case where the usual dynamical friction term [the first term on the right-hand side of (II.1)], has been replaced by $-\Lambda|p\rangle$. A statistical analysis of the solution (II.2) of the Eq. (II.1) may proceed in terms of time intervals Δt during which we can treat all functions of time except $|F(t)\rangle$ as constant. However, the time interval Δt is long enough for the position or momentum of a Brownian particle to change appreciably. It is reasonable to assume that a finite time interval $(0,t)$ can be divided into a large number of subintervals of duration Δt. Under these conditions the Eq. (II.2) can be replaced by

$$|Y\rangle=\textstyle\sum_n|Y_n\rangle, \quad (II.3)$$

where

$$|Y_n\rangle=\Phi(n\Delta t,t)|\Gamma(\Delta t)\rangle, \quad (II.4)$$

$$\Phi(\xi,t)=e^{\Lambda(\xi-t)},$$

and

$$|\Gamma(\Delta t)\rangle=\int_{n\Delta t}^{(n+1)\Delta t}|F(\xi)\rangle d\xi \quad (II.5)$$

represents the net force which may act on a Brownian particle on a given occasion during an interval of time

[3] S. Chandrasekhar, *Principles of Stellar Dynamics* (Dover Publications, Inc., New York, 1942), p. 257.

[4] S. Chandrasekhar and J. von Neumann, Astrophys. J. **95**, 489 (1942), and **97**, 1 (1943).

[5] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).

Δt. Because of the superposition argument in the next paragraph the distribution of $|Y\rangle$ to tend to a Maxwellian distribution as $t\to\infty$ the probability of occurrence of different values of $|\Gamma(\Delta t)\rangle$ must be governed by the distribution function

$$\tau(\Gamma(\Delta t)) = (1/N)\exp[-(1/4\Delta t m\kappa T) \times\langle\Gamma(\Delta t)|f^{-1}|\Gamma(\Delta t)\rangle], \quad \text{(II.6)}$$

where

$$N=\int \exp[-(1/4\Delta t m\kappa T)\langle\Gamma(\Delta t)|f^{-1}|\Gamma(\Delta t)\rangle]d^3\Gamma = [\det(4\pi m\kappa T\Delta t f)]^{1/2}. \quad \text{(II.7)}$$

The above distribution is valid only for times Δt large compared to the average period of a single fluctuation of $|F(t)\rangle$. The period of fluctuation of $|F(t)\rangle$ is of the order of the time between successive collisions between Brownian particles. The similarity of $\tau(\Gamma(\Delta t))$ with the distribution corresponding to the problem of random flights accords with the fact that the force $|\Gamma(\Delta t)\rangle$ experienced by a Brownian particle, in a time interval Δt large compared with the frequency of interparticle collisions, is a result of superposition of the large number of random forces arising from interparticle collisions. This choice of the factor $1/4m\kappa T$ is imposed by the requirement that the distribution $W(Y)$ shall tend to the Maxwellian distribution as $t\to\infty$.

With the above premises we shall prove the following lemma: For a quantity represented by a column vector $|u\rangle$ in the form

$$|u\rangle=\int_0^t \Psi(\xi,t)|F(\xi)\rangle d\xi, \quad \text{(II.8)}$$

the probability distribution is given by

$$W(\mathbf{u},\mathbf{u}_0,t)=\frac{1}{A}\exp\left[-\frac{a}{4\kappa T}\langle u|\left(\int_0^t \Psi f\tilde{\Psi}d\xi\right)^{-1}|u\rangle\right], \quad \text{(II.9)}$$

where $a=1/m$ for the distribution of momenta and $a=m$ for the distribution of positions. The normalization factor A is given by

$$A=\int \exp\left[-\frac{a}{4\kappa T}\langle u|\left(\int_0^t \Psi f\tilde{\Psi}d\xi\right)^{-1}|u\rangle\right]d^3u = \left[\det\left(\frac{4\pi\kappa T}{a}\int_0^t \Psi f\tilde{\Psi}d\xi\right)\right]^{1/2}, \quad \text{(II.10)}$$

where the tilde over Ψ implies the transposition operation. From (II.3) and (II.4) we have

$$|u\rangle=\textstyle\sum_n |u_n\rangle. \quad \text{(II.11)}$$

According to (II.6) the probability distribution of $|u_n\rangle$ is governed by

$$\eta(\mathbf{u}_n)=\frac{1}{A_n}\exp\left[-\frac{a}{4\kappa T} \times\langle u_n|(\Psi(n\Delta t)f\tilde{\Psi}(n\Delta t)\Delta t)^{-1}|u_n\rangle\right], \quad \text{(II.12)}$$

where

$$A_n=\left[\det\left(\frac{4\pi\kappa T}{a}\Psi(n\Delta t)f\tilde{\Psi}(n\Delta t)\Delta t\right)\right]^{1/2}. \quad \text{(II.13)}$$

We now use the definition of the probability distribution $W(\mathbf{u},\mathbf{u}_0,t)$ as given in Ref. 5, p. 10, in the form

$$W(\mathbf{u},\mathbf{u}_0,t)=\left(\frac{1}{2\pi}\right)^3\int \exp(-i\boldsymbol{\varrho}\cdot\mathbf{u})U(\boldsymbol{\varrho})d^3\rho, \quad \text{(II.14)}$$

where

$$U(\boldsymbol{\varrho})=\lim_{N\to\infty}\prod_{s=1}^{N}\int \eta(\mathbf{u}_s)\exp(i\boldsymbol{\varrho}\cdot\mathbf{u}_s)d^3u_s. \quad \text{(II.15)}$$

This is Markoff's method as applied to the problem of random flights.

Now, writing

$$\eta(u_s)=\frac{1}{A_s}\exp[-\langle u_s|\gamma|u_s\rangle],$$

and

$$-\langle u_s|\gamma|u_s\rangle+i\langle\rho|u_s\rangle=-\langle\chi_s|\gamma|\chi_s\rangle-\tfrac{1}{4}\langle\rho|\gamma^{-1}|\rho\rangle,$$

where

$$|\chi_s\rangle=|u_s\rangle-\tfrac{1}{2}i\gamma^{-1}|\rho\rangle,\quad \gamma=(a/4\kappa T)[\Psi(n\Delta t)f\tilde{\Psi}(n\Delta t)]^{-1},$$

we obtain

$$U(\boldsymbol{\varrho})=\lim_{N\to\infty}\prod_{s=1}^{N}\left[(1/A_s)\int \exp[-\langle\chi_s|\gamma|\chi_s\rangle]d^3\chi_s \times\exp(-\tfrac{1}{4}\langle\rho|\gamma^{-1}|\rho\rangle)\right]$$

$$=\lim_{N\to\infty}\prod_{s=1}^{N}(1/A_s)\left[\frac{\pi^3}{\det(\gamma)}\right]^{1/2}\exp[-\tfrac{1}{4}\langle\rho|\gamma^{-1}|\rho\rangle]$$

$$=\exp\left[-(\kappa T/a)\langle\rho|\left(\int_0^t \Psi f\tilde{\Psi}d\xi\right)|\rho\rangle\right]. \quad \text{(II.16)}$$

Using this result in (II.14) we get

$$W(\mathbf{u},\mathbf{u}_0,t) = (1/2\pi)^3 \int \exp(-i\boldsymbol{\varrho}\cdot\mathbf{u}) \times\exp\left[-(\kappa T/a)\langle\rho|\left(\int_0^t \Psi f\tilde{\Psi}d\xi\right)|\rho\rangle\right]d^3\rho$$

$$= \left[\frac{1}{\det\left(\frac{4\pi\kappa T}{a}\int_0^t \Psi f\tilde{\Psi}d\xi\right)}\right]^{1/2} \times\exp\left[-(a/4\kappa T)\langle u|\left(\int_0^t \Psi f\tilde{\Psi}d\xi\right)^{-1}|u\rangle\right], \quad \text{(II.17)}$$

which is the required proof for the lemma.

III. AVERAGE ENERGY

From the Eqs. (II.2) and (II.8) it follows that the operator $\Psi(\xi,t)$ is given by (II.4). We can apply the lemma above to find the probability distribution of $|Y\rangle$ as defined by (II.2). Thus,

$$\int_0^t \Psi f\tilde{\Psi}d\xi = \int_0^t \exp[\Lambda(\xi-t)]f\exp[\tilde{\Lambda}(\xi-t)]d\xi$$

$$= \frac{1}{2}\int_0^t \exp[\Lambda(\xi-t)](\Lambda+\tilde{\Lambda})\exp[\tilde{\Lambda}(\xi-t)]d\xi$$

$$= \frac{1}{2}\int_0^t \frac{d}{d\xi}\{\exp[\Lambda(\xi-t)]\exp[\tilde{\Lambda}(\xi-t)]\}d\xi$$

$$= \tfrac{1}{2}[1-\exp(-\Lambda t)\exp(-\tilde{\Lambda}t)], \quad \text{(III.1)}$$

where we put

$$f = \tfrac{1}{2}(\Lambda+\tilde{\Lambda})$$

and where the relation

$$\Lambda\tilde{\Lambda} \neq \tilde{\Lambda}\Lambda, \quad \text{(III.2)}$$

will lead to a dependence of the average energy of a particle on the external magnetic field. This is a consequence of the anisotropic dynamical friction which arises partly from the rotation of hyperbolic particle orbits by the magnetic field. This anisotropy gives rise to a coupling of the components of particle's momentum. Thus from (II.9), taking $a=1/m$, we get

$$W(\mathbf{p},\mathbf{p}_0,t) = \left[\frac{1}{\det[2\pi m\kappa T(1-\exp(-\Lambda t)\exp(-\tilde{\Lambda}t))]}\right]^{1/2} \times\exp\left[-\frac{1}{2m\kappa T}\langle Y|(1-\exp(-\Lambda t)\times\exp(-\tilde{\Lambda}t))^{-1}|Y\rangle\right]. \quad \text{(III.3)}$$

The average energy of a particle can be calculated by expressing p^2 in terms of Y as

$$p^2 = Y^2 + \langle p_0|\exp(-\tilde{\Lambda}t)\exp(-\Lambda t)|p_0\rangle + \langle p_0|\exp(-\tilde{\Lambda}t)|Y\rangle + \langle Y|\exp(-\Lambda t)|p_0\rangle.$$

The relevant integrals are of the form

$$I_1 = \int Y^2\exp[-\langle Y|\gamma|Y\rangle]d^3Y = \frac{1}{2}\left[\frac{\pi}{\det\gamma}\right]^{3/2}\sum(\text{first minors of }\gamma) = \frac{1}{2}\left[\frac{\pi}{\det\gamma}\right]^{3/2}(\gamma_1\gamma_2+\gamma_2\gamma_3+\gamma_1\gamma_3),$$

$$I_2 = \int \mathbf{Y}\exp[-\langle Y|\gamma|Y\rangle]d^3Y = 0,$$

where γ_1, γ_2, γ_3 are eigenvalues of $\gamma = (1/2m\kappa T)\times(1-\exp(-\Lambda t)\exp(-\tilde{\Lambda}t))^{-1}$. Hence,

$$\left\langle\frac{p^2}{2m}\right\rangle = \frac{1}{4m}\det[2m\kappa T(1-\exp(-\Lambda t)\exp(-\tilde{\Lambda}t))] \times\sum\text{ first minors of } \times\left[\frac{1}{2m\kappa T(1-\exp(-\Lambda t)\exp(-\tilde{\Lambda}t))}\right] + \frac{1}{2m}\langle p_0|(\exp(-\tilde{\Lambda}t)\exp(-\Lambda t))|p_0\rangle, \quad \text{(III.4)}$$

which is not independent of the external magnetic field. On this basis, one should regard the operator $\Psi f\Psi^\dagger$ as the effective dynamical friction coefficient for a plasma in a magnetic field. However, if the dynamical friction is a scalar (or diagonal), then

$$\Lambda = \beta - i\omega_c K_3 \quad\text{and}\quad \Lambda\tilde{\Lambda} = \tilde{\Lambda}\Lambda,$$

so that the distribution function reduces to

$$W = \left[\frac{1}{2\pi m\kappa T(1-e^{-2\beta t})}\right]^{3/2} \times\exp\left[-\frac{Y^2}{2m\kappa T(1-e^{-2\beta t})}\right]. \quad \text{(III.5)}$$

In this case the average energy is independent of the field,

$$\left\langle\frac{p^2}{2m}\right\rangle = \tfrac{3}{2}\kappa T(1-e^{-2\beta t}) + \frac{p_0^2}{2m}e^{-2\beta t}. \quad \text{(III.6)}$$

It is easy to verify that the distribution function (III.5) satisfies the differential equation

$$\frac{\partial W}{\partial t} = \Lambda_{ij}\frac{\partial(p_j W)}{\partial p_i} + \tfrac{1}{2}m\kappa T(\Lambda+\tilde{\Lambda})_{ij}\frac{\partial^2 W}{\partial p_i\partial p_j}, \quad \text{(III.7)}$$

where the summation convention over repeated subscripts is used. In (III.7) the first term on the right can be written as

$$\Lambda_{ij}\frac{\partial(p_jW)}{\partial p_i}=[\beta\delta_{ij}-i\omega_c(K_3)_{ij}]\frac{\partial(p_jW)}{\partial p_i}$$
$$=\beta\nabla_p\cdot(\mathbf{p}W)-\frac{e}{mc}\mathbf{p}\times\mathbf{B}\cdot\nabla_pW.$$

Hence the Eq. (III.7) becomes

$$\frac{\partial W}{\partial t}+\frac{e}{mc}\mathbf{p}\times\mathbf{B}\cdot\nabla_pW=\beta\nabla_p\cdot(\mathbf{p}W)+m\kappa T\beta\nabla_p{}^2W. \quad \text{(III.8)}$$

This is the Fokker-Planck equation in momentum space for a charge in a magnetic field. The two terms on the right-hand side arise from Brownian motion.

IV. DIFFUSION

The probability distribution function for the displacement $\mathbf{r}$ of a Brownian particle at time t given that the particle is at $\mathbf{r}_0$ with a momentum $\mathbf{p}_0$ at time $t=0$ can be obtained by using

$$|r-r_0\rangle=\frac{1}{m}\int_0^t|p\rangle dt$$

and Eq. (II.2). Thus, from Eq. (II.2) we may write

$$|R\rangle=|r-r_0\rangle-\frac{1}{m\Lambda}(1-e^{-\Lambda t})|p_0\rangle$$
$$=\frac{1}{m}\int_0^t d\rho\int_0^\rho e^{\Lambda(\xi-\rho)}|F(\xi)\rangle d\xi. \quad \text{(IV.1)}$$

The right-hand side of (IV.1) by integration by part can be written as

$$|R\rangle=\frac{1}{m}\int_0^t\frac{1}{\Lambda}[1-e^{\Lambda(\xi-t)}]|F(\xi)\rangle d\xi. \quad \text{(IV.2)}$$

Thus, in the case, to apply our lemma we must take

$$\Psi=\frac{1}{\Lambda}[1-e^{\Lambda(\xi-t)}], \quad \text{(IV.3)}$$

so that

$$\int_0^t\Psi f\tilde{\Psi}d\xi$$
$$=(\tilde{\Lambda}f^{-1}\Lambda)^{-1}t+\tfrac{1}{2}\Lambda^{-1}(1-\exp(-\Lambda t)\exp(-\tilde{\Lambda}t))(\tilde{\Lambda})^{-1}$$
$$-\tfrac{1}{2}[\Lambda^{-2}(1-\exp(-\Lambda t))+(\tilde{\Lambda})^{-2}(1-\exp(-\tilde{\Lambda}t))$$
$$+\Lambda^{-1}(2-\exp(-\Lambda t)-\exp(-\tilde{\Lambda}t))(\tilde{\Lambda})^{-1}], \quad \text{(IV.4)}$$

where we have used

$$f=\tfrac{1}{2}(\Lambda+\tilde{\Lambda}).$$

For times long compared to the order of magnitude of f^{-1} (the norm of the operator f^{-1}) we can ignore the exponential and the constant terms as compared to the first term in (IV.4), and write it as

$$\int_0^t\Psi f\tilde{\Psi}d\xi=(\tilde{\Lambda}f^{-1}\Lambda)^{-1}t. \quad \text{(IV.5)}$$

Hence, using (II.9) with $a=m$, we get

$$W(\mathbf{r},t;\mathbf{r}_0,\mathbf{p}_0)=\frac{1}{A}\exp\left[-\frac{m}{4\kappa Tt}\langle R|(\tilde{\Lambda}f^{-1}\Lambda)|R\rangle\right], \quad \text{(IV.6)}$$

where

$$A=\left[\det\left(\frac{4\pi\kappa Tt}{m}(\tilde{\Lambda}f^{-1}\Lambda)^{-1}\right)\right]^{1/2}. \quad \text{(IV.7)}$$

We may now calculate the mean square displacements across and along the magnetic field, defined by

$$\langle(\Delta R_\perp)^2\rangle=\frac{1}{2}\int[(x_1-x_{01})^2+(x_2-x_{02})^2]$$
$$\times W(\mathbf{r},t,\mathbf{r}_0,\mathbf{p}_0)d^3R, \quad \text{(IV.8)}$$

and

$$\langle(\Delta R_{\parallel})^2\rangle=\int(x_3-x_{03})^2W(\mathbf{r},t,\mathbf{r}_0,\mathbf{p}_0)d^3R, \quad \text{(IV.9)}$$

respectively. Let $1/D_1$, $1/D_2$, $1/D_3$ be the eigenvalues of $m\tilde{\Lambda}f^{-1}\Lambda/\kappa T$ corresponding to normalized eigenvectors $|\lambda_1\rangle$, $|\lambda_2\rangle$, $|\lambda_3\rangle$ which span a space with a unit operator[6]

$$I=|\lambda_1\rangle\langle\lambda_1|+|\lambda_2\rangle\langle\lambda_2|+|\lambda_3\rangle\langle\lambda_3|.$$

Hence, we may write

$$\frac{m\tilde{\Lambda}f^{-1}\Lambda}{\kappa T}=\frac{1}{D_1}|\lambda_1\rangle\langle\lambda_1|+\frac{1}{D_2}|\lambda_2\rangle\langle\lambda_2|+\frac{1}{D_3}|\lambda_3\rangle\langle\lambda_3|,$$

so that

$$\frac{m}{4\kappa Tt}\langle R|(\tilde{\Lambda}f^{-1}\Lambda)|R\rangle=\frac{u_1^2}{4D_1t}+\frac{u_2^2}{4D_2t}+\frac{u_3^2}{4D_3t},$$

where

$$u_i=\langle R|\lambda_i\rangle=\langle\lambda_i|R\rangle, \quad i=1,2,3.$$

The distribution function (IV.6) assumes the form

$$W=\left[\frac{1}{(4\pi)^3D_1D_2D_3t^3}\right]^{1/2}\exp\left[-\frac{u_1^2}{4D_1t}-\frac{u_2^2}{4D_3t}-\frac{u_3^2}{4D_3t}\right], \quad \text{(IV.10)}$$

where D_1, D_2, D_3 refer now to diffusion coefficients in the u_1, u_2, and u_3 directions, respectively.

The integrals (IV.8) and (IV.9) can easily be performed. We obtain

$$\langle(\Delta R_\perp)^2\rangle=(D_1+D_2)t \quad \text{(IV.11)}$$

[6] See B. Kurşunoğlu, *Modern Quantum Theory* (W. H. Freeman and Company, San Francisco, 1962), Chap. I.

for transverse diffusion, and

$$\langle(\Delta R_{\parallel})^2\rangle = 2D_3 t \qquad \text{(IV.12)}$$

for longitudinal diffusion.

Now in order to test the basic role of a nondiagonal element in the process of diffusion, we shall retain only one (γ) of the dynamical friction coefficients γ, δ, ρ and will assume that the other two vanish ($\delta=\rho=0$). We further set $\beta=\mu=\nu$ in the dynamical friction matrix f. With these assumptions it is quite easy to calculate the diffusion coefficients D_1, D_2, and D_3 which are just the eigenvalues of $(\kappa T/m)\ (\tilde{\Lambda} f^{-1}\Lambda)^{-1}$. These eigenvalues yield the result

$$D_1 + D_2 = \frac{2\kappa T\beta}{m(\beta^2+\omega_c^2-\gamma^2)}, \quad D_3 = \frac{\kappa T}{\beta m}, \qquad \text{(IV.13)}$$

where

$$\beta > \gamma. \qquad \text{(IV.14)}$$

The same results are obtained regardless which of the off-diagonal friction elements is retained.

We shall consider two cases:

(i) $\gamma=0$ so that

$$\langle(\Delta R_\perp)^2\rangle = \frac{2\kappa T\beta t}{m(\beta^2+\omega_c^2)} = \frac{2\kappa T t}{\beta m}\left(1 - \frac{\omega_c^2}{(\beta^2+\omega_c^2)}\right). \qquad \text{(IV.15)}$$

This is well-known classical diffusion where the ordinary Brownian motion of the particle is inhibited by the magnetic field.

(ii) $\gamma = \pm\omega_c$ yields

$$\langle(\Delta R_\perp)^2\rangle = \frac{2\kappa T t}{\beta m}. \qquad \text{(IV.16)}$$

This is an "enhanced diffusion" which takes place either at a critical value of B or for a certain value of the parameters (particle density, temperature) in γ. This result is, presumably, related to some relaxation process in the plasma placed in a magnetic field. In such a plasma relaxation times differ in different directions with respect to the direction of the magnetic field. According to (IV.16) at $\gamma=\pm\omega_c$ the diffusion process occurs as an ordinary Brownian motion uninhibited by the magnetic field.

However, it must be observed that for the enhancement of diffusion to occur we must, at least, have a plasma where the diagonal relaxation coefficient β exceeds ω_c. The latter may arise from assuming that collision frequency is much higher than ω_c, i.e.,

$$(n\langle v\sigma_c\rangle) \gg \omega_c.$$

In this case the off-diagonal relaxation term γ can be expected to increase and the condition $\gamma=\pm\omega_c$ may be realized.

V. CONCLUSION

We have shown that in the presence of an anisotropic dynamical friction force, stochastic processes in the plasma lead to an enhancement for the diffusion of particles. For a special choice ($\delta=\rho=0$) of the friction matrix, maximum diffusion independent of magnetic field (ordinary Brownian motion) sets in across the magnetic field where $\gamma=\pm\omega_c$. The significance of this result will depend very much on the form of γ as a function of particle density, temperature, and relative thermal velocities. However, despite this theoretical incompleteness it should be of great interest to look for possible experimental evidence for this type of diffusion in a plasma placed in a constant magnetic field.

This approach to the diffusion process may also be of some use in the discussion of cosmic ray accelerations. The assumption of randomly moving magnetic fields in interstellar space can form the basis for Brownian motion of cosmic rays across these magnetic clouds. The possibility of enhanced diffusion of the particles may be a reasonable mechanism for the acceleration process.

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